Logarithms and Exponents [256 marks]

1. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3 \sqrt{\cos 2x + 2}$. [3]

Markscheme

attempting to use the change of base rule *M1*

$\log_9(\cos 2x + 2) = \frac{\log_3(\cos 2x+2)}{\log_3 9}$ *A1*

$= \frac{1}{2}\log_3(\cos 2x + 2)$ *A1*

$= \log_3 \sqrt{\cos 2x + 2}$ *AG*

[3 marks]

(b) Hence or otherwise solve
$\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

Markscheme

$\log_3(2\sin x) = \log_3 \sqrt{\cos 2x + 2}$

$2\sin x = \sqrt{\cos 2x + 2}$ *M1*

$4\sin^2 x = \cos 2x + 2$ (or equivalent) *A1*

use of $\cos 2x = 1 - 2\sin^2 x$ *(M1)*

$6\sin^2 x = 3$

$\sin x = (\pm)\frac{1}{\sqrt{2}}$ *A1*

$x = \frac{\pi}{4}$ *A1*

Note: Award ***A0*** if solutions other than $x = \frac{\pi}{4}$ are included.

[5 marks]

2. [Maximum mark: 5]

Solve the equation $2 \ln x = \ln 9 + 4$. Give your answer in the form $x = p\mathrm{e}^q$ where $p,\ q \in \mathbb{Z}^+$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

METHOD 1

$2 \ln x - \ln 9 = 4$

uses $m \ln x = \ln x^m$ **(M1)**

$\ln x^2 - \ln 9 = 4$

uses $\ln a - \ln b = \ln \frac{a}{b}$ **(M1)**

$\ln \frac{x^2}{9} = 4$

$\frac{x^2}{9} = \mathrm{e}^4$ **A1**

$x^2 = 9\mathrm{e}^4 \Rightarrow x = \sqrt{9\mathrm{e}^4}\ \ (x > 0)$ **A1**

$x = 3\mathrm{e}^2\ \ (p = 3,\ q = 2)$ **A1**

METHOD 2

expresses 4 as $4\ \ln\ \mathrm{e}$ and uses $\ln\ x^m = m\ \ln\ x$ **(M1)**

$2\ \ln\ x = 2\ \ln 3 + 4\ \ln\ \mathrm{e}\ \ (\ln\ x = \ln 3 + 2\ \ln\ \mathrm{e})$ **A1**

uses $2\ \ln\ \mathrm{e} = \ln\ \mathrm{e}^2$ and $\ln\ a + \ln\ b = \ln\ ab$ **(M1)**

$\ln\ x = \ln\ \left(3\mathrm{e}^2\right)$ **A1**

$x = 3\mathrm{e}^2\ \ (p = 3,\ q = 2)$ **A1**

METHOD 3

expresses 4 as $4\ \ln\ \mathrm{e}$ and uses $m\ \ln\ x = \ln\ x^m$ **(M1)**

$\ln\ x^2 = \ln\ 3^2 + \ln\ \mathrm{e}^4$ **A1**

uses $\ln\ a + \ln\ b = \ln\ ab$ **(M1)**

$\ln\ x^2 = \ln\ \left(3^2\,\mathrm{e}^4\right)$

$x^2 = 3^2\,\mathrm{e}^4 \Rightarrow x = \sqrt{3^2\,\mathrm{e}^4}\ \ (x > 0)$ **A1**

so $x = 3\mathrm{e}^2\ \ (x > 0)\ \ (p = 3,\ q = 2)$ **A1**

[5 marks]

3. [Maximum mark: 6]

(a) Given that $x > 7$, show that $\frac{x}{x^2-8x+7} \times \frac{x^2-1}{x+1} \equiv \frac{x}{x-7}$. [2]

Markscheme

attempt to factorize at least one expression *(M1)*

$\frac{x}{(x-7)(x-1)} \times \frac{(x-1)(x+1)}{x+1}$ **OR** $\frac{x(x^2-1)}{(x-7)(x^2-1)}$ **OR** $\frac{x(x^2-1)}{(x-7)(x-1)(x+1)}$
A1

$\frac{x}{x-7}$ ***AG***

[2 marks]

(b) Hence, or otherwise, solve
$\log_2 \left[x\left(x^2-1\right)\right] - 1 = \log_2 \left[\left(x^2-8x+7\right)(x+1)\right]$
. [4]

Markscheme

METHOD 1 (combines given log terms)

attempt to use the subtraction/quotient property of logs ***(M1)***

$\log_2 \left(\frac{x(x^2-1)}{(x^2-8x+7)(x+1)}\right)$

valid attempt to rewrite equation with (or without) logs on both sides (seen anywhere) ***(M1)***

$2 = \frac{x}{x-7}$ **OR** $\log_2 \left(\frac{x}{x-7}\right) = \log_2 2$ **OR** $2^{\log_2 \left(\frac{x}{x-7}\right)} = 2^1$ ***(A1)***

$x = 14$ ***A1***

METHOD 2 (uses log laws with LHS)

attempt to use $1 = \log_2 2$ (seen anywhere) ***(M1)***

attempt to use the subtraction/quotient property of logs ***(M1)***

$\log_2 \left(\frac{x^3-x}{2}\right) = \log_2 \left(\left(x^2-8x+7\right)(x+1)\right)$ **OR**
$\frac{x(x^2-1)}{2} = (x-1)(x-7)(x+1)$ ***(A1)***

$\frac{x}{2} = x - 7$

$x = 14$ ***A1***

[4 marks]

4. [Maximum mark: 5]

Solve the equation $3 \log_8 10x - \log_4 x = 1$ for $x > 0$. [5]

Markscheme

Note: Candidates may approach this problem in different ways and may do their steps in many different orders.
The ***M*** marks are independent and may be awarded in any order. Use the description of each mark to determine when the mark may be awarded.

$3 \log_8 10x - \log_4 x = 1$

attempt to apply power rule, quotient rule, or product rule (seen anywhere) ***(M1)***

attempt to apply change of base rule (seen anywhere) ***(M1)***

correct equation with same base in all logarithms ***A1***

$\log_2 10x - \log_2 x^{\frac{1}{2}} = 1$ or
$\log_2 10 + \log_2 x - \frac{1}{2} \log_2 x = 1$ or $\frac{\log_4 (10x)^3}{\log_4 8} - \log_4 x = 1$
(or equivalent)

correct equation without logarithms ***A1***

$10\sqrt{x} = 2$ or $\sqrt{x} = \frac{2}{10} \left(= \frac{1}{5}\right)$ (or equivalent)

$x = \frac{1}{25}$ ***A1***

[5 marks]

5. [Maximum mark: 14]

A population of frogs, F, in a swamp after t months, can be modelled by the function

$F(t) = 1850 \times 1.105^t$ where $t \geq 0$.

(a) Find the population of frogs after one year. [2]

Markscheme

correct substitution of $t = 12$ into $F(t)$ ***(A1)***

$6130.82\ldots$

6130 (accept 6131) answer must be integer ***A1***

[2 marks]

(b) After x complete months, the population will be at least $35\,000$ frogs. Find the value of x. [3]

Markscheme

$1850 \times 1.105^x \geq 35\,000$ (accept strict inequality or equality) ***(A1)***

attempt to solve for x using table, graph, or logarithms ***(M1)***

$29.4471\ldots$ **OR** $(x = 29 \Rightarrow)33\,471.7\ldots$ **OR** $(x = 30 \Rightarrow)36\,986.2\ldots$

$x = 30$ (months) ***A1***

Note: Condone use of t.

[3 marks]

The function F can be written in the form $F(t) = 1850\mathrm{e}^{kt}$.

(c) Find the exact value of k. [2]

Markscheme

$\left(1.105^t = \left(\mathrm{e}^k\right)^t \Rightarrow\right)\mathrm{e}^k = 1.105$ **OR**
$\left(1.105^t = \mathrm{e}^{kt} \Rightarrow\right)kt = \ln 1.105^t$ (or equivalent) ***(A1)***

$k = \ln 1.105$ (do not accept $k = 0.0998$) ***A1***

Note: Do not award any marks if the candidate substitutes an approximate value obtained from using $F(t) = 1850 \times 1.105^t$ into $F(t) = 1850 \times \mathrm{e}^{kt}$, for example $F(12) = 1850 \times \mathrm{e}^{12k} = 6130.82\ldots$, as this will not lead to an exact value for k.

[2 marks]

(d) Find the rate at which the population of frogs is growing after 15 months. [2]

Markscheme

recognizing the need to find F' or $F'(15)$ ***(M1)***

$825.911\ldots$

$F'(15) = 826$ (frogs per month) ***A1***

[2 marks]

A more realistic model describing the population of frogs, G, after t months is given by

$G(t) = \frac{35\,000}{1+A\mathrm{e}^{-0.0998t}}$ where $t \geq 0$.

(e) After 15 months, this model predicts a population of 6995 frogs. Find the value of A.

[2]

Markscheme

setting $G(15)$ equal to 6995 ***(M1)***

$17.8890\ldots$

$A = 17.9$ ***A1***

[2 marks]

(f) Find the value of t when the rate of population growth is the greatest. [2]

Markscheme

recognize need for $G'' = 0$ or maximum of G' ***(M1)***

$28.8996\ldots$

$t = 28.9$ ***A1***

[2 marks]

(g) By considering the graph of G or otherwise, state one reason why $G(t)$ is a more appropriate long-term model than $F(t)$.

[1]

Markscheme

any one of the following reasons ***(A1)***

- $G(t)$ accounts for frog deaths in the population.

- $G(t)$ accounts for limited space/resources.

- $G(t)$ has carrying capacity.

- $F(t)$ has the population growing forever, without bound.

[1 mark]

6. [Maximum mark: 5]

Write each of the following expressions in the form $\ln\ k$, where $k \in \mathbb{Z}^+$.

(a) $\ln 3 + \ln 4$ [1]

Markscheme

$\ln\ 12$ ***A1***

[1 mark]

(b) $3 \ln 2$ [2]

Markscheme

$\ln\ 2^3$ ***(A1)***

$= \ln 8$ *A1*

[2 marks]

(c) $-\ln \frac{1}{2}$ [2]

Markscheme

$\ln \left(\frac{1}{2}\right)^{-1}$ **OR** $-(\ln 1 - \ln 2)$ **OR** $-\ln 2^{-1}$ *(A1)*

$= \ln 2$ *A1*

[2 marks]

7. [Maximum mark: 15]

The function f is defined by $f(x) = 5(x+1)(x+3)$, where $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a\left(x - h^2\right) + k$, where $a, h, k \in \mathbb{Z}$. [4]

Markscheme

METHOD 1

$a = 5$ *(A1)*

attempt to use roots and symmetry to find h *(M1)*

$h = \frac{(-1)+(-3)}{2}$ **OR** half the distance between the roots $\frac{(-1)-(-3)}{2} = 1$ (may be seen on a diagram)

$h = -2$ (accept $x = -2$) *(A1)*

$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ *A1*

$(a = 5,\ h = -2,\ k = -5)$

METHOD 2

$a = 5$ ***(A1)***

attempt to expand

$(x+1)(x+3) = x^2 + 4x + 3$ **OR**
$5(x+1)(x+3) = 5x^2 + 20x + 15$

EITHER

uses their expansion to attempt to complete the square to the form ***(M1)***

$p(x+q)^2 + r$, where q is half the coefficient of their x term

$= (x+2)^2 - 2^2 + 3\left(= (x+2)^2 - 1\right)$ **OR**
$5\left[(x+2)^2 - 2^2 + 3\right]\left(= 5(x+2)^2 - 5\right)$ ***(A1)***

OR

uses their expansion to attempt to differentiate and sets equal to zero ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4 = 0$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x + 20 = 0$

$h = -2$ (accept $x = -2$) ***(A1)***

OR

uses their expansion to attempt to find axis of symmetry using $h = \frac{-b}{2a}$ *(M1)*

$h = \frac{-4}{2}$ **OR** $h = \frac{-20}{10}$

$h = -2$ (accept $x = -2$) *(A1)*

THEN

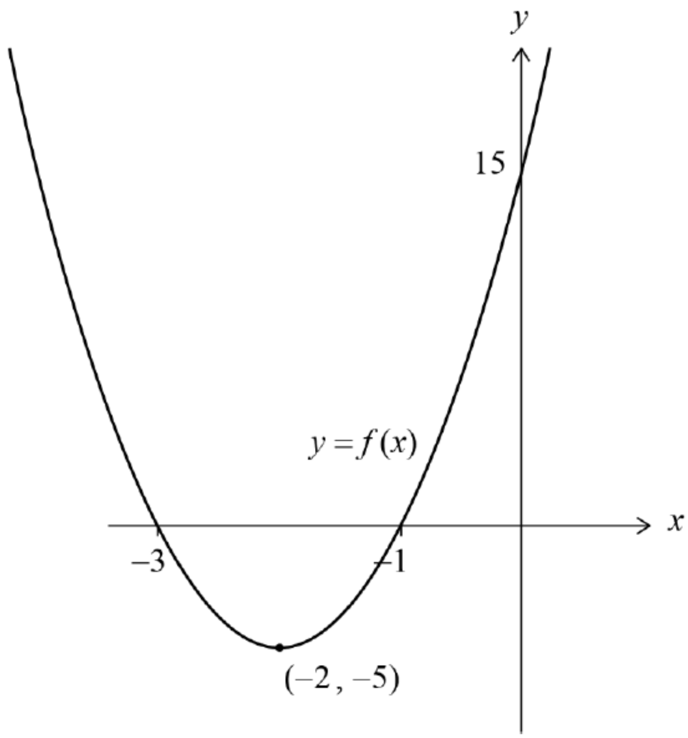
$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

$(a = 5,\ h = -2,\ k = -5)$

[4 marks]

(b) Sketch the graph of $y = f(x)$, showing the values of any intercepts with the axes and the coordinates of the vertex. [4]

Markscheme



M1A1A1A1

award ***M1*** for a roughly symmetric curve which is concave up

award ***A1*** for x intercepts at -3 and -1

award ***A1*** for y intercept at 15

award ***A1*** for vertex at $(-2, -5)$

[4 marks]

(c) Solve the inequality $f(x) \leq 40$. [4]

Markscheme

$5(x+2)^2 - 5 \leq 40$ **OR** $5(x+1)(x+3) \leq 40$ **OR**
$(x+1)(x+3) \leq 8$ leading to $(x+2)^2 \leq 9$ **OR**
$5x^2 + 20x - 25 \leq 0$ **OR** $x^2 + 4x - 5 \leq 0$ ***(A1)***

valid attempt to find the critical values for their quadratic inequality ***(M1)***

$x+2=\pm 3$ **OR** $(x+5)(x-1)=0$

$x=-5, x=1$ ***(A1)***

$-5 \leq x \leq 1$ ***A1***

Note: Accept $(x \in)[-5, 1]$ or equivalent.

[4 marks]

The function g is defined by $g(x)=\ln x$, where $x \in \mathbb{R}, x>0$.

(d.i) Write down an expression for $(f \circ g)(x)$. [1]

Markscheme

$(f \circ g)(x)=5(\ln x+1)(\text{In } x+3)$ **OR** $5(\ln x+2)^2-5$ **OR** $5(\ln x)^2+20\ln x+15$ ***A1***

[1 mark]

(d.ii) Solve the inequality $(f \circ g)(x) \leq 40$. [2]

Markscheme

attempt to replace x with $\ln x$ using their solution to part (c) ***(M1)***

$-5 \leq \ln x \leq 1$

$\mathrm{e}^{-5} \leq x \leq \mathrm{e}$ ***A1***

Note: Accept $(x \in)\left[\mathrm{e}^{-5}, \mathrm{e}\right]$ or equivalent.

[2 marks]

8. [Maximum mark: 17]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$.

(a) Find $f^{-1}(8)$. Express your answer in the form $\frac{p}{q}$ where $p,\ q \in \mathbb{Z}$. [3]

Markscheme

$4^x = 8$ **OR** $f^{-1}(x) = \log_4 x$ **OR** $f^{-1}(8) = \log_4 8$ ***(A1)***

attempt to use indices with same base **OR** change of base of logs ***(M1)***

$2^{2x} = 2^3,\ 4^x = 4^{\frac{3}{2}}$ **OR** $f^{-1}(8) = \log_4 4^{\frac{3}{2}}$ **OR**
$f^{-1}(8) = \frac{\log_2 8}{\log_2 4}$

$f^{-1}(8) = \frac{3}{2}$ ***A1***

[3 marks]

The function g is defined by $g(x) = 1 + \log_2 x$, where $x \in \mathbb{R}^+$.

(b.i) Find an expression for $g^{-1}(x)$. [2]

Markscheme

interchanging x and y (seen anywhere) ***(M1)***

$x = 1 + \log_2 y$ **OR** $y - 1 = \log_2 x$

$x - 1 = \log_2 y$ **OR** $2^{y-1} = x$

$g^{-1}(x) = 2^{x-1}$ (or equivalent) ***A1***

[2 marks]

(b.ii) Describe a sequence of transformations that transforms the graph of $y = g^{-1}(x)$ to the graph of $y = f(x)$. [2]

Markscheme

METHOD 1

a horizontal translation/shift by 1 unit to the left (do not accept 'move')

followed by a horizontal stretch/dilation with scale factor $\frac{1}{2}$ (accept horizontal compression by a factor of 2)

one correct transformation ***A1***

two correct transformations in the correct order ***A1***

METHOD 2

horizontal stretch/dilation with scale factor $\frac{1}{2}$ (accept horizontal compression by a factor of 2) ***A1***

vertical stretch/dilation with scale factor 2 ***A1***

Note: ***A1***s can be awarded independently here as order is not important.

[2 marks]

(c) Show that $(f \circ g)(x) = 4x^2$. [3]

Markscheme

attempt to find composite function (in any order) ***M1***

$f(1+\log_2 x)\ \left(=4^{1+\log_2 x}\right)$

$4\times 4^{\log_2 x}$ **OR** $4\times 2^{2\log_2 x}$ **OR** $4^{\log_2 2x}$ **OR** $2^{(2+2\log_2 x)}$ **OR** $4^{(\log_4 4+2\log_4 x)}$ ***(A1)***

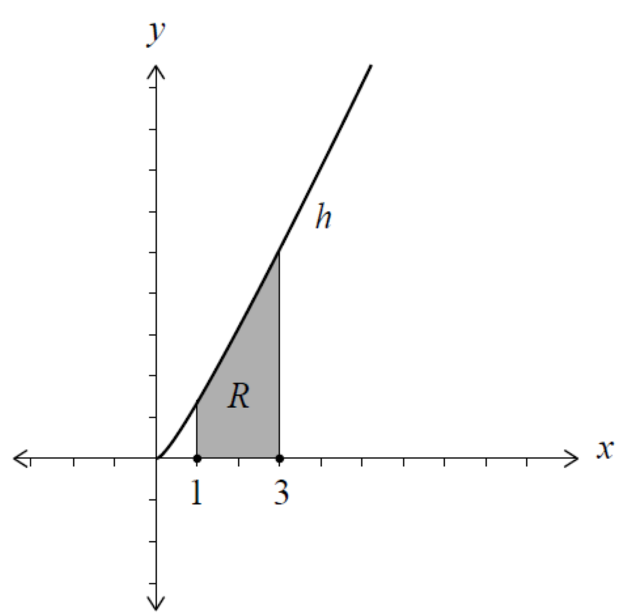
$=4\times 2^{\log_2 x^2}$ **OR** $2^{\log_2 (2x)^2}$ **OR** $4^{\log_4 (4x^2)}$ ***A1***

$=4x^2$ ***AG***

[3 marks]

The function h is defined by $h(x)=\frac{4x^2}{2x+1},\ x\neq -\frac{1}{2}$.

The following diagram shows part of the graph of h. Let R be the region enclosed by the graph of h and the x-axis, between the lines $x=1$ and $x=3$
.



(d.i) Show that $2x-1+\frac{1}{2x+1}=\frac{4x^2}{2x+1}$. [2]

Markscheme

$2x-1+\frac{1}{2x+1}=\frac{(2x-1)(2x+1)+1}{2x+1}$ ***(A1)***

$=\frac{4x^2-1+1}{2x+1}$ ***A1***

$2x-1+\frac{1}{2x+1}=\frac{4x^2}{2x+1}$ ***AG***

Note: accept working from RHS to LHS.

[2 marks]

(d.ii) Hence or otherwise, find the area of R, giving your answer in the form $p+q\ln r$, where $p,\ q,\ r\in\mathbb{Q}^+$. [5]

Markscheme

METHOD 1

enclosed area is $\int_1^3\left(2x-1+\frac{1}{2x+1}\right)\mathrm{d}\,x$

attempt to integrate (at least one correct term or $\ln\,(2x+1)$ seen)
(M1)

$=x^2-x+\frac{1}{2}\ln\,|2x+1|(+c)$ ***A1A1***

Note: Award ***A1*** for x^2-x and ***A1*** for $\frac{1}{2}\ln\,|2x+1|$.
Accept $\frac{1}{2}\ln\,(2x+1)$.

substitute correct limits into their integrated expression and subtract
(M1)

$= \left(9 - 3 + \frac{1}{2}\ln 7\right) - \left(1 - 1 + \frac{1}{2}\ln 3\right)$

$A = 6 + \frac{1}{2}\ln \frac{7}{3}$ ***A1***

METHOD 2

attempt to use integration by substitution ***(M1)***

let $u = 2x + 1$ **OR** $u = 2x - 1 \Rightarrow \frac{du}{dx} = 2$

$= \frac{1}{2}\int \left(u - 2 + \frac{1}{u}\right) \mathrm{d}\, u$ **OR** $\frac{1}{2}\int \frac{(u-1)^2}{u} \mathrm{d}\, u$ **OR**
$\frac{1}{2}\int \left(u + \frac{1}{u+2}\right) \mathrm{d}\, u$ ***A1***

correct integration

$= \frac{1}{4}u^2 - u + \frac{1}{2}\ln |u| (+c)$ **OR** $\frac{1}{4}u^2 + \frac{1}{2}\ln |u+2| (+c)$ ***A1***

substitution of their limits into their integrated expression and subtract
(M1)

$= \left(\frac{49}{4} - 7 + \frac{1}{2}\ln 7\right) - \left(\frac{9}{4} - 3 + \frac{1}{2}\ln 3\right)$ **OR**
$\left(\frac{25}{4} + \frac{1}{2}\ln 7\right) - \left(\frac{1}{4} + \frac{1}{2}\ln 3\right)$

$A = 6 + \frac{1}{2}\ln \frac{7}{3}$ (or equivalent) ***A1***

[5 marks]

9. [Maximum mark: 5]
Let $\log_{10} 2 = p$ and $\log_{10} 3 = q$.

(a) Find an expression for $\log_{10} 24$ in terms of p and q. [3]

Markscheme

correct application of $\log_a xy = \log_a x + \log_a y$ **OR** $\log_a x^m = m \log_a x$ ***(M1)***

correct expression in terms of $\log_{10} 2$ **AND** $\log_{10} 3$ (arguments must be 2 and 3) ***(A1)***

$3 \log_{10} 2 + \log_{10} 3$ **OR** $\log_{10} 2 + \log_{10} 2 + \log_{10} 2 + \log_{10} 3$

$3p + q$ ***A1***

[3 marks]

(b) Find an expression for $\log_3 8$ in terms of p and q. [2]

Markscheme

$(\log_3 8 =)\frac{\log_{10} 8}{\log_{10} 3}\left(= \frac{3 \log_{10} 2}{\log_{10} 3}\right)$ ***(A1)***

$= \frac{3p}{q}$ ***A1***

[2 marks]

10. [Maximum mark: 16]

(a.i) Solve $5 - 4x - x^2 = 0$. [2]

Markscheme

recognizing need to factorise, complete the square or substitute into quadratic formula ***(M1)***

$(5+x)(1-x)$ **OR** $9-(x+2)^2$ **OR** $\frac{4 \pm \sqrt{(-4)^2 - 4 \times (-1) \times 5}}{2 \times (-1)}$

$x = -5,\ x = 1$ ***A1***

[2 marks]

(a.ii) Hence, find the values of x such that $5 - 4x - x^2 > 0$. [2]

Markscheme

sign diagram or sketch of $y = 5 - 4x - x^2$ with -5 and 1 indicated ***(M1)***

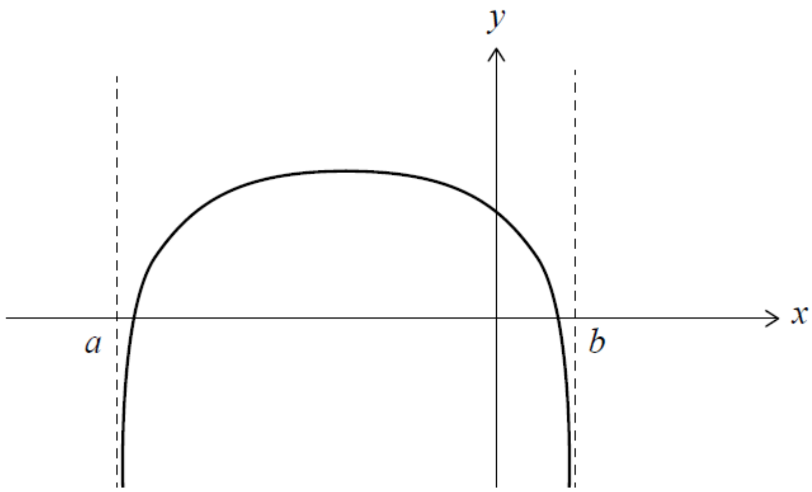
$-5 < x < 1$ ***A1***

Note: Award ***A1*** for answers using interval notation $(-5,\ 1)$.

[2 marks]

Consider the function $f(x) = \log_k (5 - 4x - x^2)$, where $a < x < b$ and $k > 1$.

Part of the graph of f is shown in the following diagram.



The graph of f has vertical asymptotes at $x = a$ and $x = b$.

(b) Write down the value of

(b.i) a; [1]

Markscheme

$(a =) - 5$ ***A1***

[1 mark]

(b.ii) b. [1]

Markscheme

$(b =)1$ ***A1***

[1 mark]

(c) Find the exact values of x such that $f(x) = 0$. [4]

Markscheme

$\left(\log_k\left(5 - 4x - x^2\right) = 0 \Rightarrow\right)5 - 4x - x^2 = 1$ ***(A1)***

$x^2 + 4x - 4 = 0$ **OR** $4 - 4x - x^2 = 0$

attempting to solve their quadratic equation set to 0 ***(M1)***

$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times (-4)}}{2 \times 1}$ **OR** $(x+2)^2 - 8\ (= 0)$ ***A1***

$x = -2 \pm 2\sqrt{2}$ ***A1***

[4 marks]

The graph of f has a maximum value of 2.

(d) Find the value of k. [6]

Markscheme

EITHER

recognizing that maximum of f is when $5 - 4x - x^2$ is a maximum or when $f\prime(x) = 0$ ***(M1)***

substituting into $x = -\frac{b}{2a}$ **OR** $-4 - 2x = 0$ **OR** $\frac{-4-2x}{(5-4x-x^2)\ln k} = 0$

OR

use of symmetry to find x-coordinate of maximum ***(M1)***

$\frac{a+b}{2}$ **OR** $x = \frac{\text{sum of their answers to (c)}}{2}$

THEN

$x = -2$ (seen anywhere) ***A1***

maximum of quadratic is 9 **OR** argument of logarithm is 9 ***(A1)***

$\log_k (9) = 2$ **OR** $\log_k (9) = \log_k \left(k^2\right)$ **OR**
$k^2 = 5 - 4(-2) - (-2)^2$ (or equivalent) ***A1***

valid approach to solve ***(M1)***

$k^2 = 9$ **OR** $k = \pm 3$

$k = 3$ ***A1***

[6 marks]

11. [Maximum mark: 14]
The function f is defined as $f(x) = \log_2 (4x)$, where $x > 0$.

(a) Find the value of

(a.i) $f(8)$; [2]

Markscheme

substitution of $x = 8$ in part (i) or $x = \frac{1}{4}$ in part (ii) ***(M1)***

$\log_2 (4 \times 8)$ OR $\log_2 (32)$

$\log_2 (32) = 5$ ***A1***

[2 marks]

(a.ii) $f\left(\frac{1}{4}\right)$. [1]

Markscheme

$f\left(\frac{1}{4}\right) = 0$ ***A1***

[1 mark]

(b) Find an expression for $f^{-1}(x)$. [4]

Markscheme

swap x and y ***(M1)***

$x = \log_2(4y)$ OR $x = 2 + \log_2 y$

attempt to write as exponential ***(M1)***

$2^x = 4y$ ***(A1)***

$\left(f^{-1}(x) =\right) \frac{2^x}{4} \quad \left(= 2^{x-2}\right)$ ***A1***

[4 marks]

(c) Hence, or otherwise, find $f^{-1}(0)$. [1]

Markscheme

$\frac{1}{4}$ ***A1***

[1 mark]

The graph of $y = f\left(16x^3\right)$ can be obtained by translating and stretching the graph of $y = \log_2 x$.

(d) Describe these two transformations specifying the order in which they are to be applied. [6]

Markscheme

METHOD 1

$f\left(16x^3\right) = \log_2 \left(4 \times 16x^3\right)$

attempt to use addition rule for logs ***(M1)***

$\log_2 4 + \log_2 16 + \log_2 \ x^3$ OR $\log_2 64 + \log_2 x^3$ (or equivalent) ***(A1)***

attempt to use exponent property for logarithms ***(M1)***

$f\left(16x^3\right) = 6 + 3 \log_2 x$ (or equivalent) ***A1***

the graph of g must be vertically stretched (dilated) by a scale factor of 3 and then vertically translated (shifted) 6 units upwards. ***A2***

METHOD 2 (5 marks maximum)

$f\left(16x^3\right) = \log_2 \left(4 \times 16x^3\right)$

attempt to write argument as a power ***(M1)***

$\log_2 \left(64x^3\right) = \log_2 \left(\left(4x\right)^3\right)$ (or equivalent) ***(A1)***

attempt to use exponent property for logarithms ***(M1)***

$f\left(16x^3\right) = 3 \log_2 \left(4x\right)$ (or equivalent) ***A1***

EITHER

the graph of g must be vertically stretched (dilated) by a scale factor of 3 and stretched (dilated) horizontally by a scale factor of $\frac{1}{4}$. ***A1***

OR

the graph of g must be stretched (dilated) horizontally by a scale factor of $\frac{1}{4}$ and vertically stretched (dilated) by a scale factor of 3. ***A1***

Note: In this method, the final mark is ***A1***, as the question specifically asks for a translation and a stretch.

[6 marks]

12. [Maximum mark: 5]

Consider the function $h(x) = \log_{10}\left(3x^2 - rx + r - 2\right)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

Markscheme

METHOD 1

recognition that $3x^2 - rx + r - 2$ must be greater than zero (seen anywhere) ***R1***

(discriminant $=$) $(-r)^2 - 4(3)(r-2) \quad \left(= r^2 - 12r + 24\right)$ (seen anywhere) ***(A1)***

$2.53589\ldots \quad \left(= 6 - 2\sqrt{3}\right)$ AND

$9.46410\ldots \quad \left(= 6 + 2\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $3x^2 - rx + r - 2$ is less than zero ***(M1)***

$2.54 < r < 9.46 \quad \left(6 - 2\sqrt{3} < r < 6 + 2\sqrt{3}\right)$ ***A1***

Note: Accept $2.54 \leq r \leq 9.46$.

METHOD 2

recognition that $3x^2 - rx + r - 2$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when
$x = \frac{r}{6} \Rightarrow \left(y =\right) 3\left(\frac{r}{6}\right)^2 - r\left(\frac{r}{6}\right) + r - 2 \ \left(> 0\right)$ ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{3x^2-2}{x-1}$ OR $x > 1 \Rightarrow r < \frac{3x^2-2}{x-1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{3x^2-2}{x-1}$ ***(M1)***

THEN

$\left(r >\right) 2.53589\ldots \quad \left(= 6 - 2\sqrt{3}\right)$ AND
$\left(r <\right) 9.46410\ldots \quad \left(= 6 + 2\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$2.54 < r < 9.46 \quad \left(6 - 2\sqrt{3} < r < 6 + 2\sqrt{3}\right)$ ***A1***

Note: Accept $2.54 \leq r \leq 9.46$.

[5 marks]

13. [Maximum mark: 5]
Consider the function $h(x) = \log_{10}\left(4x^2 - rx + r - 1\right)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

Markscheme

METHOD 1

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

(discriminant $=$) $(-r)^2 - 4(4)(r-1) \quad \left(= r^2 - 16r + 16\right)$ (seen anywhere) ***(A1)***

$1.07179\ldots \quad \left(= 8 - 4\sqrt{3}\right)$ AND
$14.9282\ldots \quad \left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $4x^2 - rx + r - 1$ is less than zero ***(M1)***

$1.07 < r < 14.9 \quad \left(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3}\right)$ ***A1***

Note: Accept $1.08 \leq r \leq 14.9$.

METHOD 2

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when
$x = \frac{r}{8} \Rightarrow \left(y =\right) 4\left(\frac{r}{8}\right)^2 - r\left(\frac{r}{8}\right) + r - 1 \ \left(> 0\right)$ ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{4x^2-1}{x-1}$ OR $x > 1 \Rightarrow r < \frac{4x^2-1}{x-1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{4x^2-1}{x-1}$ ***(M1)***

THEN

$\left(r >\right) 1.07179\ldots \quad \left(= 8 - 4\sqrt{3}\right)$ AND
$\left(r <\right) 14.9282\ldots \quad \left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$1.07 < r < 14.9 \quad \left(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3}\right)$ ***A1***

Note: Accept $1.08 \leq r \leq 14.9$.

[5 marks]

14. [Maximum mark: 5]
It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10} \left(\frac{1}{a}\right)$; [2]

Markscheme

$\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$= -\frac{1}{3}$ *A1*

[2 marks]

(b) $\log_{1000} a$. [3]

Markscheme

$\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3} \log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x \left(= \left(10^3\right)^x\right)$ *(A1)*

$\frac{\log_{10} a}{3}$ OR $\frac{1}{3} \log_{1000} 1000^{\frac{1}{3}}$ OR $\log_{1000} 1000^{\frac{1}{9}}$ OR $3x = \frac{1}{3}$ *(A1)*

$= \frac{1}{9}$ *A1*

[3 marks]

15. [Maximum mark: 16]

Consider the arithmetic sequence $a,\ p,\ q \ldots,$ where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Markscheme

attempt to find a difference *(M1)*

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation *A1*

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

Consider the geometric sequence $a,\ s,\ t\ldots$, where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

Markscheme

attempt to find a ratio ***(M1)***

$r = \frac{s}{a},\ r^2 = \frac{t}{a},\ r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Markscheme

EITHER

$2p - 1 = s^2$ (or equivalent) ***A1***

$(s^2 > 0) \Rightarrow 2p - 1 > 0$ OR $s = \sqrt{2p-1} \Rightarrow 2p - 1 > 0$ OR $p = \frac{s^2+1}{2}$ (and $s^2 > 0$) ***R1***

OR

$2p-1=a$ and $s^2=a$ ***A1***

($s^2>0,$ so) $a>0 \Rightarrow 2p-1>0$ OR $p\frac{a+1}{2}$ and $a>0$ ***R1***

$\Rightarrow p>\frac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

Consider the case where $a=9,\ s>0$ and $q=t=1$.

(d) Write down the first four terms of the

(d.i) arithmetic sequence; [2]

Markscheme

$9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2^{nd} term and 4^{th} term

[2 marks]

(d.ii) geometric sequence. [2]

Markscheme

$9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3,$ and $u_3 = 1 + \ln 1.$

(e.i) Find the common difference of the new sequence in terms of $\ln 3.$ [3]

Markscheme

attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR
$d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2 \ln 3$ OR $\ln 1 = 0$ OR
$\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

[3 marks]

(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3.$ [3]

Markscheme

METHOD 1

attempt to substitute first term and their common difference into S_{10}

(M1)

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR
$\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ (or equivalent) ***A1***

$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

METHOD 2

$u_{10} = 9 + \ln 9 + 9(-4 - \ln 3)(= -27 + \ln 9 - 9 \ln 3)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR
$\frac{10}{2}(9 + \ln 9 - 27 + \ln 9 - 9 \ln 3)$ OR

$\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ OR
$\frac{10}{2}(9 + \ln 9 - 27 - 7 \ln 3)$ (or equivalent) ***A1***

$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

[3 marks]

16. [Maximum mark: 12]

Consider the function defined by $f(x) = \frac{3}{2}e^{x-2},\ 0 \le x \le 4.$

(a) Show that the inverse function is given by
$f^{-1}(x) = 2 + \ln\left(\frac{2x}{3}\right).$

[3]

Markscheme

METHOD 1

attempt to interchange x and y ***M1***

Note: This ***M1*** may be awarded at any stage in the working.

attempt to rearrange using definition of natural log or take the natural log of both sides ***M1***

$\frac{2x}{3} = \mathrm{e}^{y-2} \Rightarrow \ln\left(\frac{2x}{3}\right) = y - 2$ OR
$x = \frac{3}{2}\mathrm{e}^{y-2} \Rightarrow \ln(x) = \ln\left(\frac{3}{2}\right) + y - 2$ ***A1***

$y = 2 + \ln\left(\frac{2x}{3}\right)$

so $f^{-1}(x) = 2 + \ln\left(\frac{2x}{3}\right)$ ***AG***

METHOD 2

attempt to verify that $\left(f \circ f^{-1}\right)(x) = x$ ***M1***

$\left(f \circ f^{-1}\right)(x) = \frac{3}{2}\mathrm{e}^{\ln\left(\frac{2x}{3}\right)+2-2}\left(= \frac{3}{2}\mathrm{e}^{\ln\left(\frac{2x}{3}\right)}\right)$

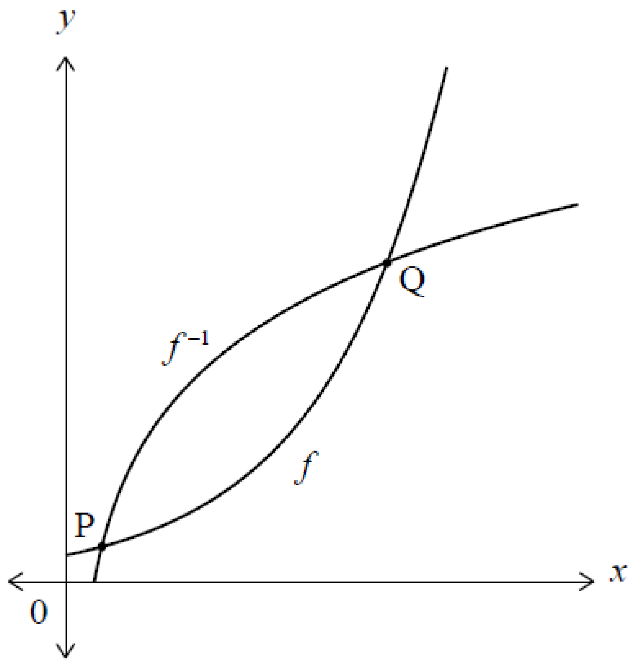
attempt to use definition of natural log ***M1***

$\left(f \circ f^{-1}\right)(x) = \frac{3}{2} \times \frac{2x}{3}$ ***A1***

$\left(f \circ f^{-1}\right)(x) = x$ ***AG***

[3 marks]

The graphs of f and f^{-1} intersect at two points P and Q, as shown on the following diagram.



(b) Find PQ. [3]

Markscheme

$(0.264456\ldots, 0.264456\ldots)$ AND $(2.51799\ldots, 2.51799\ldots)$ ***(A1)***

Note: Award ***A1*** for $0.264456\ldots$ and $2.51799\ldots$ seen.

attempt to put their values in distance formula or use of the isosceles right-angled triangle ***(M1)***

$$\sqrt{(2.51799\ldots - 0.264456\ldots)^2 + (2.51799\ldots - 0.264456\ldots)^2}$$

OR

$\sqrt{2} \times (2.51799\ldots - 0.264456\ldots)$

$= 3.18689\ldots$

$= 3.19$ ***A1***

[3 marks]

The graph of f is reflected in the x-axis and then translated parallel to the y-axis by 5 units in the positive direction to give the graph of a function g.

(c) Write down

(c.i) an expression for $g(x)$; [2]

Markscheme

$g(x) = -\frac{3}{2}\mathrm{e}^{x-2} + 5$ OR $g(x) = -f(x) + 5$ ***A1A1***

Note: Award ***A1*** for each correct term.

[2 marks]

(c.ii) the domain of g. [1]

Markscheme

$0 \leq x \leq 4$ ***A1***

[1 mark]

(d) Solve the equation $f(x) = g(x)$. Give your answer in the form $x = a + \ln b$, where $a,\ b \in \mathbb{Q}$. [3]

Markscheme

$\frac{3}{2}\mathrm{e}^{x-2} = -\frac{3}{2}\mathrm{e}^{x-2} + 5$ OR $f(x) = -f(x) + 5$

attempt to collect together terms in e^{x-2} or $f(x)$ ***(M1)***

$3\mathrm{e}^{x-2}$ OR $2f(x) = 5$

$\mathrm{e}^{x-2} = \frac{5}{3}$ OR $x = f^{-1}\left(\frac{5}{2}\right)$ ***(A1)***

$x = 2 + \ln\left(\frac{5}{3}\right)$ ***A1***

$\left(a = 2,\ b = \frac{5}{3}\right)$

Note: Award ***A1*** for each correct term given in exact form.

[3 marks]

17. [Maximum mark: 6]

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10\log_{10}\left(I \times 10^{12}\right)$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

Markscheme

$I = 2 \times 10^{-6}\left(= \frac{1}{500\,000}\right)$ (units) ***A1***

[1 mark]

(b) Determine the loudness of S_2. [2]

Markscheme

substitutes their doubled I-value from part (a) into L ***(M1)***

$L = 10\log_{10}\left(2 \times 10^{-6} \times 10^{12}\right)\ (= 63.0102\ldots)$

$= 63.0$ (decibels) ***A1***

Note: Accept $60 + 10\log_{10} 2$ (decibels) as a final answer.

Do not award the final ***A1*** for $L = 0$ (from $I = 10^{-12}$).

[2 marks]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

Markscheme

$115 = 10\log_{10}\left(I \times 10^{12}\right)$ ***(A1)***

attempts to solve for I ***(M1)***

$I = \frac{10^{11.5}}{10^{12}}$ (or equivalent) $(= 0.316227\ldots)$

$I = 0.316$ (units) ***A1***

Note: Accept exact final answers such as $10^{-0.5}$ and $\frac{1}{\sqrt{10}}$.

[3 marks]

18. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 7)$, where $x > \frac{7}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0,\ d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

Markscheme

$x = 0$ ***A1***

[1 mark]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b.i) Show that, at the points of intersection, $x^2 - 2dx + 7d = 0$. [4]

Markscheme

setting $\ln(2x - 7) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln (2x-7) = \ln \frac{x^2}{d}$ OR $\ln \frac{x^2}{2x-7} = \ln d$ OR
$\ln (2x-7)d = \ln x^2$

$\frac{x^2}{d} = 2x - 7$ OR $\frac{x^2}{2x-7} = d$ OR $(2x-7) = x^2$ ***A1***

$x^2 - 2dx + 7d = 0$ ***AG***

[4 marks]

(b.ii) Hence, show that $d^2 - 7d > 0.$ [3]

Markscheme

discriminant = $(-2d)^2 - 4 \times 7d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(2d)^2 - 4 \times 7d > 0$ OR $4d^2 - 28d > 0$ ***A1***

$d^2 - 7d > 0$ ***AG***

[3 marks]

(b.iii) Find the range of possible values of *d*. [2]

Markscheme

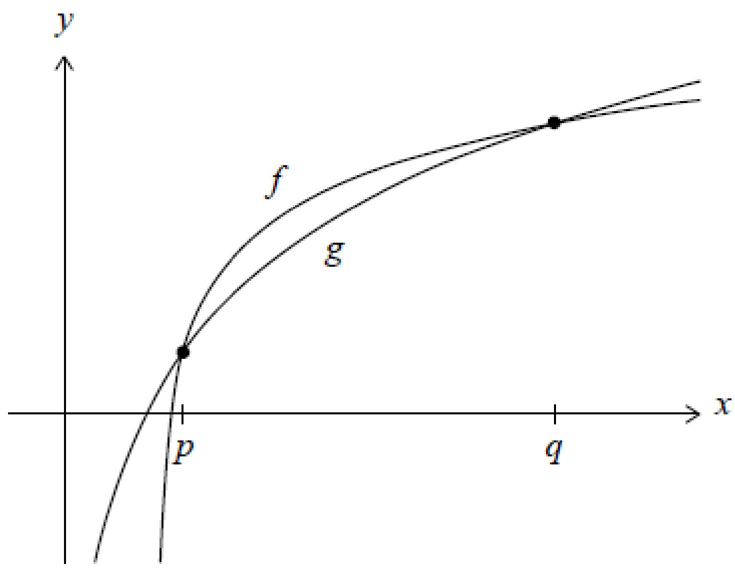
setting $d(d-7) > 0$ OR $d(d-7) = 0$ OR sketch graph OR sign test OR $d^2 > 7d$ ***(M1)***

$d < 0$ or $d > 7,$ but $d \in \mathbb{R}^+$

$d > 7$ (or $]7, \infty[$) ***A1***

[2 marks]

The following diagram shows parts of the graph $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a,\ b \in \mathbb{Z}^+$. [5]

Markscheme

$x^2 - 20x + 70\ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 30 = 0\right)$ or $\left((\mathrm{x} =)\frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 70}}{2}\right)$

$x = 10 - \sqrt{30}(= p)$ or $x = 10 + \sqrt{30}(= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{30}\ \left(\text{or}\ \sqrt{120}\right)$ **A1**

$(a = 2,\ b = 30)$ (or $a = 1,\ b = 120$)

[5 marks]

19. [Maximum mark: 6]

Find the range of possible values of k such that $\mathrm{e}^{2x} + \ln k = 3\mathrm{e}^{x}$ has at least one real solution. [6]

Markscheme

recognition of quadratic in e^{x} ***(M1)***

$(\mathrm{e}^{x})^2 - 3\mathrm{e}^{x} + \ln k\left(= 0\right)$ OR $A^2 - 3A + \ln k(= 0)$

recognizing discriminant ≥ 0 (seen anywhere) ***(M1)***

$(-3)^2 - 4\left(1\right)\left(\ln k\right)$ OR $9 - 4\ln k$ ***(A1)***

$\ln k \leq \frac{9}{4}$ ***(A1)***

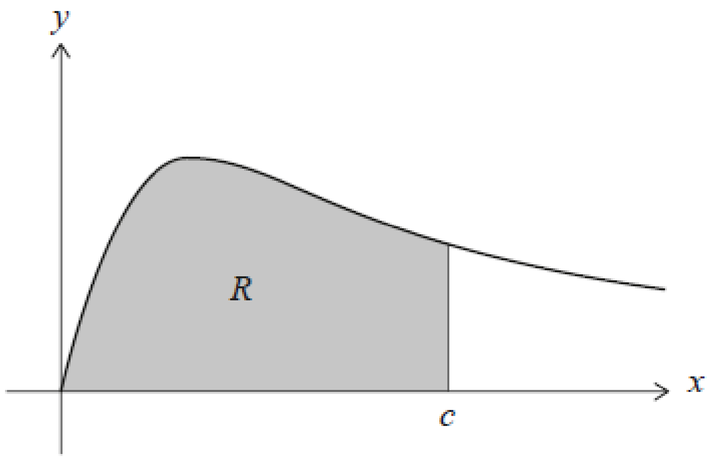
$\mathrm{e}^{9/4}$ (seen anywhere) ***A1***

$0 < k \leq \mathrm{e}^{9/4}$ ***A1***

[6 marks]

20. [Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2+2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$. [6]

Find the value of c.

Markscheme

$$A = \int_0^c \frac{x}{x^2+2} \mathrm{d}x$$

EITHER

attempts to integrate by inspection or substitution using $u = x^2 + 2$ or $u = x^2$ ***(M1)***

Note: If candidate simply states $u = x^2 + 2$ or $u = x^2$, but does not attempt to integrate, do not award the ***(M1)***.

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k \ln u$ or $k \ln(u + 2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln\left(u+2\right)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k\ \ln\left(x^2+2\right)$.

$\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln\left(c^2+2\right)-\ln 2\right)\left(=\ln 3\right)$ OR
$\frac{1}{2}\ln\left(c^2+2\right)-\frac{1}{2}\ln 2\left(=\ln 3\right)$

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)\left(=\ln 3\right)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}\left(=\ln 3\right)$ OR
$\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln\left(c^2+2\right)-\ln 2-\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}\left(=\ln 3\right)$ OR $\ln\sqrt{\frac{c^2+2}{\sqrt{2}}}\left(=\ln 3\right)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ **OR** $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c=\pm 4$ as a final answer.

[6 marks]

21. [Maximum mark: 15]

Calculate the value of each of the following logarithms:

(a.i) $\log_2 \frac{1}{16}$. [2]

Markscheme

valid approach to find the required logarithm ***(M1)***

$2^x=\frac{1}{16}$ OR $2^x=2^{-4}$ OR $\frac{1}{16}=2^{-4}$ OR $\log_2 1-\log_2 16$

$\log_2 \frac{1}{16} = -4$ ***A1***

[2 marks]

(a.ii) $\log_9 3$. [2]

Markscheme

valid approach to find the required logarithm ***(M1)***

$9^x = 3$ OR $3^{2x} = 3$ OR $3 = 9^{\frac{1}{2}}$ OR $\frac{\log_3 3}{\log_3 9}$

$\log_9 3 = \frac{1}{2}$ ***A1***

[2 marks]

(a.iii) $\log_{\sqrt{3}} 81$. [3]

Markscheme

$\left(\sqrt{3}\right)^x = 81$ OR $\frac{\log_3 81}{\log_3 \sqrt{3}}$ ***(A1)***

$(3)^{\frac{x}{2}} = 3^4$ OR $\frac{x}{2} = 4$ OR $\frac{4}{\frac{1}{2}}$ ***(A1)***

$x = 8$ ***A1***

[3 marks]

It is given that $\log_{ab} a = 3$, where $a,\ b \in \mathbb{R}^+,\ ab \neq 1$.

(b.i) Show that $\log_{ab} b = -2$. [4]

Markscheme

Note: There are many valid approaches to the question, and the steps may be seen in different ways. Some possible methods are given here, but candidates may use a combination of one or more of these methods.

In all methods, the final ***A*** mark is awarded for working which leads directly to the ***AG***.

METHOD 1

$(ab)^3 = a$ ***(A1)***

attempt to isolate b or a power of b ***(M1)***

correct working ***(A1)***

$b = \frac{a}{a^3b^2}$ OR $b^3 = a^{-2}$ OR $b^{-1} = (ab)^2$ OR $b^3 = \frac{1}{a^2}$

$b = \frac{1}{a^2b^2}$ OR $b = (ab)^{-2}$ OR $3\log_{ab} b = -2\log_{ab} a$ OR $-\log_{ab} b = 2\log_{ab} ab$ ***A1***

$\log_{ab} b = -2$ ***AG***

METHOD 2

$(ab)^3 = a$ ***(A1)***

taking logarithm to base ab on both sides ***(M1)***

$\log_{ab} (ab)^3 = \log_{ab} a$ OR $\log_{ab} a^3b^3 = \log_{ab} a$

correct application of log rules leading to equation in terms of $\log_{ab}$ ***(A1)***

$3\log_{ab} a + 3\log_{ab} b = \log_{ab} a$ OR $3\log_{ab} b = -2\log_{ab} a$ OR $\log_{ab} b^3 = \log_{ab} a^{-2}$

$\log_{ab} b = \log_{ab} a^{-\frac{2}{3}}$ OR $\log_{ab} b = -\frac{2}{3}\log_{ab} a$ OR $\log_{ab} b = -\frac{2}{3}(3)$ ***A1***

$\log_{ab} b = -2$ ***AG***

Note: Candidates may substitute $\log_{ab} a = 3$ at any point in their working.

METHOD 3

$\log_{ab} a = 3$

writing in terms of base a ***(M1)***

$\frac{\log_a a}{\log_a ab}(= 3)$

correct application of log rules ***(A1)***

$\frac{\log_a a}{\log_a a + \log_a b}(= 3)$ OR $\frac{1}{1 + \log_a b}(= 3)$ OR $3\log_a b = -2$ OR $\log_a b = -\frac{2}{3}$

writing $\log_{ab} b$ in terms of base a ***(A1)***

$\log_{ab} b = \frac{\log_a b}{\log_a a + \log_a b}$

correct working ***A1***

$\log_{ab} b = \frac{-\frac{2}{3}}{1 - \frac{2}{3}}$ OR $\frac{\left(-\frac{2}{3}\right)}{\left(\frac{1}{3}\right)}$

$\log_{ab} b = -2$ ***AG***

METHOD 4

$\log_{ab} ab = 1$ ***A2***

$\log_{ab} a + \log_{ab} b = 1$ ***(A1)***

$3 + \log_{ab} b = 1$ ***A1***

$\log_{ab} b = -2$ ***AG***

[4 marks]

(b.ii) Hence find the value of $\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}$. [4]

Markscheme

applying the quotient rule or product rule for logs

$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}} = \log_{ab} \sqrt[3]{a} - \log_{ab} \sqrt{b}$ OR
$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}} = \log_{ab} \sqrt[3]{a} + \log_{ab} \frac{1}{\sqrt{b}}$ ***(A1)***

correct working ***(A1)***

$= \frac{1}{3}\log_{ab} a - \frac{1}{2}\log_{ab} b$ OR $\log_{ab} ab - \log_{ab} \sqrt{b}$

$= \frac{1}{3} \cdot 3 - \frac{1}{2}(-2)$ ***(A1)***

$= 2$ ***A1***

Note: Award ***A1A0A0A1*** for a correct answer with no working.

[4 marks]

22. [Maximum mark: 5]

(a) The expression $\frac{3\sqrt{x}-5}{\sqrt{x}}$ can be written as $3 - 5x^p$. Write down the value of p. [1]

Markscheme

$\frac{3\sqrt{x}-5}{\sqrt{x}} = 3 - 5x^{-\frac{1}{2}}$ ***A1***

$p = -\frac{1}{2}$

[1 mark]

(b) Hence, find the value of $\int_1^9 \left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right) \text{ d } x$. [4]

Markscheme

$\int \frac{3\sqrt{x}-5}{\sqrt{x}} \text{ d } x = 3x - 10x^{\frac{1}{2}}(+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9) - 10(9)^{\frac{1}{2}} - \left(3(1) - 10(1)^{\frac{1}{2}}\right)$ OR
$27 - 10 \times 3 - (3 - 10)$

$= 4$ ***A1***

[4 marks]

23. [Maximum mark: 5]

Consider the curve with equation $y = \left(2x - 1\right)\mathrm{e}^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5\mathrm{e}^{k}x$.

Find the value of k. [5]

Markscheme

evidence of using product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 1) \times \left(k\mathrm{e}^{kx}\right) + 2 \times \mathrm{e}^{kx} \quad \left(= \mathrm{e}^{kx}(2kx - k + 2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1 \Rightarrow k\mathrm{e}^{k} + 2\mathrm{e}^{k}$

OR

slope of tangent is $5\mathrm{e}^{k}$

their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ equals the *slope* of $y = 5\mathrm{e}^{k}x \quad \left(= 5\mathrm{e}^{k}\right)$ (seen anywhere) ***(M1)***

$k\mathrm{e}^{k} + 2\mathrm{e}^{k} = 5\mathrm{e}^{k}$

$k = 3$ ***A1***

[5 marks]

24. [Maximum mark: 15]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm \frac{1}{\sqrt{3}}$. [2]

Markscheme

EITHER

attempt to use a ratio from consecutive terms ***M1***

$\frac{p \ln x}{\ln x} = \frac{\frac{1}{3} \ln x}{p \ln x}$ OR $\frac{1}{3} \ln x = (\ln x) r^2$ OR $p \ln x = \ln x \left(\frac{1}{3p} \right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} \ldots$ and consider the powers of x in geometric sequence

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm \frac{1}{\sqrt{3}}$ ***A1***

$p = \pm \frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

[2 marks]

(a.ii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Markscheme

$\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR
$\ln x = 3 - \sqrt{3} + \sqrt{3} - 1 \ (\Rightarrow \ln x = 2)$ ***A1***

$x = \mathrm{e}^2$ ***A1***

[3 marks]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

Markscheme

METHOD 1

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p \ln x - \ln x = \frac{1}{3}\ln x - p \ln x$ OR
$\frac{1}{3}\ln x = \ln x + 2(p \ln x - \ln x)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$

$2p \ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p \ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p \ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \ \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p \ln x - \ln x = -\frac{1}{3}\ln x$
A1

$p \ln x = \frac{2}{3} \ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

[3 marks]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

Markscheme

$d = -\frac{1}{3} \ln x$ ***A1***

[1 mark]

(b.iii) The sum of the first n terms of the series is $-3 \ln x$.

Find the value of n. [6]

Markscheme

METHOD 1

$S_n = \frac{n}{2}\left[2 \ln x + (n-1) \times \left(-\frac{1}{3} \ln x\right)\right]$

attempt to substitute into S_n and equate to $-3 \ln x$ ***(M1)***

$\frac{n}{2}\left[2 \ln x + (n-1) \times \left(-\frac{1}{3} \ln x\right)\right] = -3 \ln x$

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2 \ln x - \frac{n}{3} \ln x + \frac{1}{3} \ln x\right]$ OR $n \ln x - \frac{n(n-1)}{6} \ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right) \ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3}-\frac{n}{3}\right)=-3$ OR $n-\frac{n(n-1)}{6}=-3$ or equivalent

Note: Award as above if the series $1+p+\frac{1}{3}+\ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3}-\frac{n}{3}\right)=-3$.

attempt to form a quadratic $=0$ ***(M1)***

$n^2-7n-18=0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2)=0$

$n=9$ ***A1***

METHOD 2

listing the first 7 terms of the sequence ***(A1)***

$\ln x+\frac{2}{3}\ln x+\frac{1}{3}\ln x+0-\frac{1}{3}\ln x-\frac{2}{3}\ln x-\ln x+\ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} term $=-3\ln x$ ***(A1)***

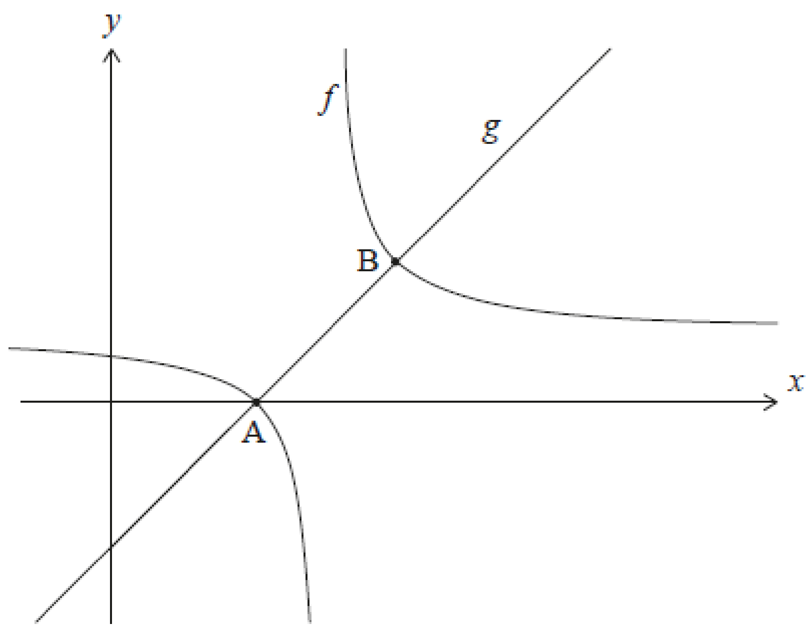
$n=9$ ***A1***

[6 marks]

25. [Maximum mark: 15]

Consider the functions $f(x) = \frac{1}{x-4} + 1$, for $x \neq 4$, and $g(x) = x - 3$ for $x \in \mathbb{R}$.

The following diagram shows the graphs of f and g.



The graphs of f and g intersect at points A and B. The coordinates of A are $(3,\ 0)$.

(a) Find the coordinates of B. [5]

Markscheme

$\frac{1}{x-4} + 1 = x - 3$ ***(M1)***

$x^2 - 8x + 15 = 0$ OR $(x-4)^2 = 1$ ***(A1)***

valid attempt to solve **their** quadratic ***(M1)***

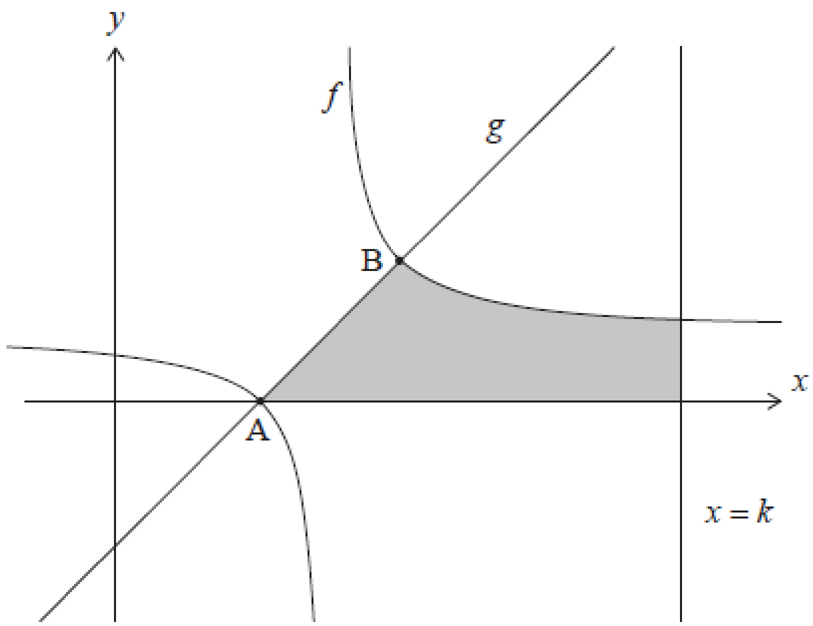
$(x-3)(x-5) = 0$ OR $x = \frac{8 \pm \sqrt{8^2 - 4(1)(15)}}{2(1)}$ OR $(x-4) = \pm 1$

$x = 5\ \ (x = 3,\ x = 5)$ (may be seen in answer) ***A1***

$\mathrm{B}(5,\ 2)$ (accept $x = 5,\ y = 2$) ***A1***

[5 marks]

In the following diagram, the shaded region is enclosed by the graph of f, the graph of g, the x-axis, and the line $x = k$, where $k \in \mathbb{Z}$.



The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

(b) Find the value of k and the value of p. [10]

Markscheme

recognizing two correct regions from $x = 3$ to $x = 5$ and from $x = 5$ to $x = k$ ***(R1)*** triangle $+\int_5^k f(x)\,\mathrm{d}\,x$ OR $\int_3^5 g(x)\,\mathrm{d}\,x+\int_5^k f(x)\,\mathrm{d}\,x$ OR

$\int_3^5 (x-3)\,\mathrm{d}\,x+\int_5^k \left(\frac{1}{x-4}+1\right)\,\mathrm{d}\,x$

area of triangle is 2 OR $\frac{2\cdot 2}{2}$ OR $\left(\frac{5^2}{2}-3(5)\right)-\left(\frac{3^2}{2}-3(3)\right)$

(A1)

correct integration ***(A1)(A1)***

$\int\left(\frac{1}{x-4}+1\right)\,\mathrm{d}\,x=\ln(x-4)+x\ (+C)$

Note: Award ***A1*** for $\ln(x-4)$ and ***A1*** for x.
Note: The first three ***A*** marks may be awarded independently of the ***R*** mark.

substitution of **their** limits (for x) into **their** integrated function (in terms of x) ***(M1)***

$\ln(k-4)+k-(\ln 1+5)$

$[\ln(x-4)+x]_5^k=\ln(k-4)+k-5$ ***A1***

adding **their** two areas (in terms of k) and equating to $\ln p+8$ ***(M1)***

$2+\ln(k-4)+k-5=\ln p+8$

equating **their** non-log terms to 8 (equation must be in terms of k) ***(M1)***

$k-3=8$

$k=11$ ***A1***

$11-4=p$

$p=7$ ***A1***

[10 marks]

26. [Maximum mark: 15]

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8$. [2]

Markscheme

$f\left(\frac{2}{3}\right) = 4$ OR $a^{\frac{2}{3}} = 4$ ***(M1)***

$a = 4^{\frac{3}{2}}$ OR $a = \left(2^2\right)^{\frac{3}{2}}$ OR $a^2 = 64$ OR $\sqrt[3]{a} = 2$ ***A1***

$a = 8$ ***AG***

[2 marks]

(b) Write down an expression for $f^{-1}(x)$. [1]

Markscheme

$f^{-1}(x) = \log_8 x$ ***A1***

Note: Accept $f^{-1}(x) = \log_a x$.
Accept any equivalent expression for f^{-1} e.g. $f^{-1}(x) = \frac{\ln x}{\ln 8}$.

[1 mark]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Markscheme

correct substitution *(A1)*

$\log_8 \sqrt{32}$ OR $8^x = 32^{\frac{1}{2}}$

correct working involving log/index law *(A1)*

$\frac{1}{2}\log_8 32$ OR $\frac{5}{2}\log_8 2$ OR $\log_8 2 = \frac{1}{3}$ OR $\log_2 2^{\frac{5}{2}}$ OR $\log_2 8 = 3$ OR $\frac{\ln 2^{\frac{5}{2}}}{\ln 2^3}$ OR $2^{3x} = 2^{\frac{5}{2}}$

$f^{-1}\left(\sqrt{32}\right) = \frac{5}{6}$ *A1*

[3 marks]

Consider the arithmetic sequence $\log_8 27$, $\log_8 p$, $\log_8 q$, $\log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

Markscheme

METHOD 1

equating a pair of differences *(M1)*

$u_2 - u_1 = u_4 - u_3 (= u_3 - u_2)$

$\log_8 p - \log_8 27 = \log_8 125 - \log_8 q$

$\log_8 125 - \log_8 q = \log_8 q - \log_8 p$

$\log_8\left(\frac{p}{27}\right) = \log_8\left(\frac{125}{q}\right), \ \log_8\left(\frac{125}{q}\right) = \log_8\left(\frac{q}{p}\right)$ ***A1A1***

$\frac{p}{27} = \frac{125}{q}$ and $\frac{125}{q} = \frac{q}{p}$ ***A1***

$27, \ p, \ q$ and 125 are in geometric sequence ***AG***

Note: If candidate assumes the sequence is geometric, award no marks for part (i). If $r = \frac{5}{3}$ has been found, this will be awarded marks in part (ii).

METHOD 2

expressing a pair of consecutive terms, in terms of d ***(M1)***

$p = 8^d \times 27$ and $q = 8^{2d} \times 27$ OR $q = 8^{2d} \times 27$ and $125 = 8^{3d} \times 27$

two correct pairs of consecutive terms, in terms of d ***A1***

$\frac{8^d \times 27}{27} = \frac{8^{2d} \times 27}{8^d \times 27} = \frac{8^{3d} \times 27}{8^{2d} \times 27}$ (must include 3 ratios) ***A1***

all simplify to 8^d ***A1***

$27, \ p, \ q$ and 125 are in geometric sequence ***AG***

[4 marks]

(d.ii) Find the value of p and the value of q. [5]

Markscheme

METHOD 1 (geometric, finding r)

$u_4 = u_1 r^3$ OR $125 = 27(r)^3$ *(M1)*

$r = \frac{5}{3}$ (seen anywhere) ***A1***

$p = 27r$ OR $\frac{125}{q} = \frac{5}{3}$ *(M1)*

$p = 45,\ q = 75$ ***A1A1***

METHOD 2 (arithmetic)

$u_4 = u_1 + 3d$ OR $\log_8 125 = \log_8 27 + 3d$ *(M1)*

$d = \log_8\left(\frac{5}{3}\right)$ (seen anywhere) ***A1***

$\log_8 p = \log_8 27 + \log_8\left(\frac{5}{3}\right)$ OR
$\log_8 q = \log_8 27 + 2\log_8\left(\frac{5}{3}\right)$ *(M1)*

$p = 45,\ q = 75$ ***A1A1***

METHOD 3 (geometric using proportion)

recognizing proportion *(M1)*

$pq = 125 \times 27$ OR $q^2 = 125p$ OR $p^2 = 27q$

two correct proportion equations ***A1***

attempt to eliminate either p or q *(M1)*

$q^2 = 125 \times \frac{125\times27}{q}$ OR $p^2 = 27 \times \frac{125\times27}{p}$

$p = 45,\ q = 75$ ***A1A1***

[5 marks]